Solutions
Math 220
HW # 4
October 8, 2018

**Exercise 1.** *Prove that if $x$ is an odd integer, then $9x+5$ is even.*

*Proof.* Suppose that $x$ is odd, then $x = 2n+1$ for some $n \in \mathbb{Z}$. Then

$$9x+5 = 9(2n+1)+5 = 18n+9+5 = 2(9n+7).$$

Since $9n+7 \in \mathbb{Z}$, $9x+5$ is even. $\square$

**Exercise 2.** *Prove that if $a$ and $c$ are odd integers, then $ab+bc$ is even for every integer $b$.*

*Proof.* Suppose that $a$ and $c$ are odd so that there are integers $k, l \in \mathbb{Z}$ such that $a = 2k+1$ and $c = 2l+1$. Then

$$ab+bc = (2k+1)b + b(2l+1) = 2kb+b+2lb+b = 2kb+2lb+2b = 2(kb+lb+b).$$

So since $kb+lb+b \in \mathbb{Z}$, $ab+bc$ is even. $\square$

**Exercise 3.** *Prove that if $n \in \mathbb{Z}$, then $n^2-3n+9$ is odd.*

*Proof.* We prove this in cases:

<u>$n$ is even</u> Since $n$ is even, $n = 2k$ for some $k \in \mathbb{Z}$. Then

$$n^2-3n+9 = 4k^2-6k+9 = 2(2k^2-3k+4)+1$$

and therefore $n^2-3n+9$ is odd.

<u>$n$ is odd</u> If $n$ is odd, then $n = 2k+1$ for some integer $k$. So

$$n^2-3n+9 = (4k^2+4k+1)-(6k+3)+9 = 4k^2-2k+7 = 2(2k^2-k+3)+1$$

verifying that $n^2-3n+9$ is odd.

$\square$

**Exercise 4.** *Let $x \in \mathbb{Z}$. Prove that if $7x+5$ is odd, then $x$ is even.*

*Proof.* We prove this by contrapositive. Suppose that $x$ is odd so that $x = 2k+1$ for some integer $k$. Then

$$7x+5 = 7(2k+1)+5 = 14k+12 = 2(7k+6)$$

showing that $7x+5$ is even. The desired conclusion follows from contrapositive. $\square$

**Exercise 5.** *Let $n \in \mathbb{Z}$. Prove that $(n+1)^2 - 1$ is even if and only if $n$ is even.*
*Proof.*

($\Longleftarrow$) Suppose that $n$ is even. Then $n = 2k$ for an some integer $k$. Then

$$(n+1)^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 2(2k^2 + 2k)$$

which is even.

($\Longrightarrow$) We prove this direction by contrapositive. Assume $n$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$(n+1)^2 - 1 = (2k+1+1)^2 - 1 = (2k+2)^2 - 1 = 4(k+1)^2 - 2 + 1 = 2(2(k+1)^2 - 1) + 1$$

which shows that $(n+1)^2 - 1$ is odd.

□

**Exercise 6.** *Disprove the statement: If $n \in \{1, 2, 3, 4, 5\}$, then $2n^2 + 1$ is divisible by 3.*

*Proof.* Let $n = 3$. Then $2n^2 + 1 = 2(3)^2 + 1 = 19$ which is not divisible by 3. Thus $n = 3$ is a counterexample. □

**Exercise 7.** *Prove that 200 cannot be written as the sum of an odd integer and two even integers.*

*Proof.* Suppose that 200 can be written as the sum of an odd integer and two even integers. Suppose that $200 = a + b + c$ where $a$ is odd and $b$ and $c$ are even. Then $b + c$ is even since both $b$ and $c$ are even. Since the sum of an odd and even integer is odd, we have $a + b + c$ is odd. Therefore 200 must be odd, a contradiction. Therefore 200 cannot be written as the sum of an odd integer and two even integers. □

**Exercise 8.** *Prove that if $n$ is an odd integer, then $7n - 5$ is even by (a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.*
*Proof.*

(a) Suppose that $n$ is odd, so $n = 2k + 1$ for some integer $k$. Then

$$7n - 5 = 7(2k+1) - 5 = 14k + 7 - 5 = 14k + 2 = 2(7k+1)$$

showing that $7n - 5$ is even.

(b) Suppose that $7n - 5$ is odd. Then $(7n - 5) + 5 = 7n$ must be even since it is the sum of two odd integers. Because 7 is odd, in order for $7n$ to be even, we must have that $n$ is even. The desired result follows by taking the contrapositive.

(c) Suppose that $n$ is odd and $7n - 5$ is also odd. Since $n$ is odd, $7n$ is also odd. Then $(7n - 5) - 7n = -5$ is even since it is the difference of two odd numbers. However $-5$ is not even, so we have our contradiction.

□

**Exercise 9.** *Prove for every integer $n \geq 8$ that there exist nonnegative integers $a$ and $b$ such that $n = 3a + 5b$.*

*Proof.* We divide this into 3 cases:

$\underline{n = 3k}$ In this case, let $a = k$ and $b = 0$, then $n = 3k + 5 \cdot 0$, as desired.

$\underline{n = 3k + 1}$ In this case, we must have $n \geq 10$, so $k \geq 3$. Then

$$n = 3k + 1 = 3(k - 3) + 9 + 1 = 3(k - 3) + 5 \cdot 2$$

so $a = k - 3$ and $b = 2$ are the desired nonnegative integers.

$\underline{n = 3k + 2}$ In this case, we must have $n \geq 8$, so $k \geq 2$. Then

$$n = 3k + 1 = 3(k - 1) + 3 + 2 = 3(k - 1) + 5 \cdot 1$$

so $a = k - 1$ and $b = 1$ are the desired nonnegative integers.

□

**Exercise 10.** *Disprove the statement: There is an integer $n$ such that $n^4 + n^3 + n^2 + n$ is odd.*

*Proof.* If $n$ is even, then $n$, $n^2$, $n^3$, and $n^4$ are all even, so $n^4 + n^3 + n^2 + n$ is even. If $n$ is odd, $n$, $n^2$, $n^3$, and $n^4$ are all odd. Since the sum of two odd integers is even, and we are adding an even number of terms, all of which are odd, the sum $n^4 + n^3 + n^2 + n$ is even. Therefore the statement is false. □